W35. Let $S_n(x)$ be polynomial defined by recurrence

$$S_{n+1} - 2(x+1)S_n + S_{n-1} = 2x, n \in \mathbb{N}$$

with initial conditions $S_0 = 0, S_1 = x.$ Prove that

$$S_n(x) \leq (1+nx)^n - 1, x \geq 0, n \in \mathbb{N};$$

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Solution by proposer.

Let $T_n(x) := S_n(x-1) + 1$ then $S_n(x-1) \leq (1+n(x-1))^n - 1, x \geq 1, n \in \mathbb{N} \Leftrightarrow$

(1) $T_n(x) \leq (1+n(x-1))^n, x \geq 1, n \in \mathbb{N}$

and $S_{n+1}(x-1) - 2xS_n(x-1) + S_{n-1}(x-1) = 2(x-1) \Leftrightarrow$

$T_{n+1}(x) - 1 - 2x(T_n(x) - 1) + T_n(x) - 1 = 2(x-1) \Leftrightarrow$

$T_{n+1}(x) = 2x \cdot T_n(x) - T_{n-1}(x), n \in \mathbb{N}.$

Since $S_0(x) = 0$ and $S_1(x) = x \Leftrightarrow S_0(x-1) = 0$ and $S_1(x-1) = x-1$

then $T_0(x) = 1$ and $T_1(x) = S_1(x-1) + 1 = x - 1 + 1 = x.$

That is $T_n(x)$ is the First Kind Chebyshev's Polynomial.

Since $(1+n(x-1))^n = \sum_{k=0}^{n} n^k \binom{n}{k} (x-1)^k$ then for the proof of inequality (**1**)

convenient to use Taylor's representation of $T_n(x)$:

$T_n(x) = \sum_{k=0} \frac{T_n^{(k)}(1)}{k!}(x-1)^k,$ because suffice to prove that $\frac{T_n^{(k)}(1)}{k!} \leq \binom{n}{k} n^k \Leftrightarrow$

(2) $T_n^{(k)}(1) \leq n^k \cdot \frac{n!}{(n-k)!}$ where $k = 0, 1, \ldots, n.$

For calculation $T_n^{(k)}(1)$ we will partake derivative equation which define

$n-th$ Chebyshev's Polynomial $T_n(x)$:

(3) $(1-x^2)T_n''(x) - xT_n'(x) + n^2 T_n(x) = 0.$

In the supposition that for arbitrary $k = 1, 2, \ldots, n-1$ consecutive derivatives

$T_n^{(k+1)}(x), T_n^{(k)}(x), T_n^{(k-1)}$ satisfy to correlation

$(1-x^2)T_n^{(k+1)}(x) - a_k x T_n^{(k)}(x) + b_k T_n^{(k-1)}(x) = 0$ we obtain that

$-2xT_n^{(k+1)}(x) + (1-x^2)T_n^{(k+2)}(x) - a_k T_n^{(k)}(x) - a_k x T_n^{(k+1)}(x) + b_k T_n^{(k)}(x) = 0 \Leftrightarrow$

$(1-x^2)T_n^{(k+2)}(x) - (a_k + 2)xT_n^{(k+1)}(x) + (b_k - a_k)T_n^{(k)}(x) = 0.$

Thus we have $a_{k+1} = a_k + 2$ and $b_{k+1} = b_k - a_k$ where $a_1 = 1$ and $b_1 = n^2.$

Hence, $a_k = 2k - 1$ and $b_{k+1} - b_1 = \sum_{i=1}^{k}(b_{i+1} - b_i) = -\sum_{i=1}^{k}(2i-1) = -k^2$ and

therefore $a_{k+1} = 2k+1, b_{k+1} = n^2 - k^2.$

So, for $T_n^{(k+2)}(x), T_n^{(k+1)}(x), T_n^{(k)}$ we obtain following correlation

(4) $(1-x^2)T_n^{(k+2)}(x) - (2k+1)xT_n^{(k+1)}(x) + (n^2 - k^2)T_n^{(k)}(x) = 0,$ where

$k = 0, 1, \ldots, n-1$ and in particularly for $x = 1$ we have:

(5) $(2k+1)T_n^{(k+1)}(1) = (n^2 - k^2)T_n^{(k)}(1) \Leftrightarrow \frac{T_n^{(k+1)}(1)}{T_n^{(k)}(1)} = \frac{n^2 - k^2}{2k+1}.$

For $k = 0$ inequality (2) obviously holds because $T_n^{(0)}(1) = T_n(1) = 1.$

Since $\frac{T_n^{(k+1)}(1)}{T_n^{(k)}(1)} \leq \frac{n^{k+1} \cdot \frac{n!}{(n-k-1)!}}{n^k \cdot \frac{n!}{(n-k)!}} \Leftrightarrow \frac{n^2-k^2}{2k+1} \leq n(n-k) \Leftrightarrow$

$n + k \leq (2k+1)n$ then from of Math. Induction's supposition that

$$T_n^{(k)}(1) \leq n^k \cdot \frac{n!}{(n-k)!}$$

we immediately obtain that $T_n^{(k+1)}(1) = T_n^{(k)}(1) \cdot \frac{T_n^{(k+1)}(1)}{T_n^{(k)}(1)} \leq$

$$n^k \cdot \frac{n!}{(n-k)!} \cdot n(n-k) = n^{k+1} \cdot \frac{n!}{(n-(k+1))!}.$$

Equality in (1) occurs iff $n = 1$ and don't holds if $n > 1$.(because

$T_n(x) = (1+n(x-1))^n, x \geq 1 \Leftrightarrow T_n^{(k)}(1) = n^k \cdot \frac{n!}{(n-k)!}, k = 0,1,\ldots,n \Leftrightarrow T_n(1) = 1$

and $\frac{T_n^{(k+1)}(1)}{T_n^{(k)}(1)} = n(n-k), k = 1,\ldots,n-1 \Leftrightarrow \frac{n^2-k^2}{2k+1} = n(n-k), k = 1,\ldots,n-1 \Leftrightarrow$

$n + k = (2k+1)n, k = 1,\ldots,n-1 \Leftrightarrow (2n-1)k = 0$).